DIFFERENTIAL EQUATIONS

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Def^n: An eq^n involving independant variable, dependant variable & the diff. coeff. of dep. var. w.r.t. indep. var.

eg $\frac{dy}{dx} + y = xe^x$

- Order – Highest order of diff. coeff appearing in DE.

- Degree – Exponent of the highest order diff. coeff, when DE is expressed as a polynomial in all the diff. coeffs

(i.e given DE is made free from all radicals & fractions as power of diff. coeff.)

$$f_n(x,y)\frac{d^n y}{dx^n} + f_{(n-1)}(x,y)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \dots + f_0(x,y) = 0$$

NOTE: If DE is not expressible in the given form, then $\nexists$ degree and the degree is said to be undefined..

Q. Find order & degree of D.E

	O	D
(i) $\left(\frac{d^3y}{dx^3}\right)^{2/3} = \left(\frac{dy}{dx}\right)+2$	3	2
(ii) $\frac{d^2y}{dx^2} = x\,l\left(\frac{dy}{dx}\right)$	2	X
(iii) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx}+3}$	2	3
(iv) $\frac{d^3y}{dx^3} = s\left(\frac{dy}{dx}\right)$	3	X
(v) $\frac{dy}{dx} = \sqrt{3x+5}$	1	1
☆ (vi) $y = 1 + \frac{dy}{dx} + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \ldots$	1	1

$\Rightarrow \quad y = e^{\left(\frac{dy}{dx}\right)} \quad \Rightarrow \quad \frac{dy}{dx} = l(y)$

FORMATION OF DE

To form DE for a given eqⁿ, we first need to identify the # independent parameters

(Parameters that cannot be clubbed)

eg. -
$$y = \lambda_1 e^{\lambda_2 x} + \lambda_3 e^{(x+\lambda_4)}$$
$$= \lambda_1 e^{\lambda_2 x} + \underbrace{\lambda_3 e^{\lambda_4}}_{\lambda} e^{x}$$
$$= \lambda_1 e^{\lambda_2 x} + \lambda e^{x}$$

It seems that this eqn has 4 indep. pars.,
but actually, it has only 3!

We can differentiate a given eqn max. 'n' times, if there are 'n' indep. pars to obtain its D.E.
We then need to replace the indep. pars using the given eqn & those we have obtained by differentiation.

* DE is devoid of all indep. pars.

Q Construct DE

(i) $y^2 = 4a(x+b)$ (ii) $xy = ae^{x} + be^{-x}$

(iii) $c(y+c)^2 = x^3$ (iv) $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$

(v) $(a+bx)e^{y/x} = x$ (vi) $y = (x-k)^2$

(vii) DE of all parabolas with axes ∥ to x-axis & having LR = a

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A. (i) $y^2 = 4ax + ab = Ax + B$

$\frac{d}{dx}$

$\Rightarrow 2yy' = A \xrightarrow{\frac{d}{dx}} \underline{yy'' + (y')^2 = 0}$

(ii) $xy = ae^x + be^{-x} \overset{\frac{d}{dx}}{\Rightarrow} xy' + y = ae^x - be^{-x}$

$\frac{d}{dx}$ $\quad xy'' + 2y' = ae^x + be^{-x}$

$\Rightarrow \underline{xy'' + 2y' = xy}$

(iii) $cy^2 + 2c^2y + c^3 = x^3$

$\frac{d}{dx}$

$\Rightarrow 2cyy' + 2c^2y' = 3x^2$

$\Rightarrow 2y'(cy + c^2) = 3x^2$

$\Rightarrow \left(\frac{2xy'}{3}\right)(cy + c^2) = x^3$

$= cy^2 + 2c^2y + c^3$

$= y(cy + c^2) + c(cy + c^2)$

$\Rightarrow \frac{2xy'}{3} = (y + c)$

$\left(\frac{2xy'}{3} - y\right) = c$

$\underline{\left(\frac{2xy'}{3} - y\right)\left(\frac{2xy'}{3}\right)^2 = x^3}$

(iv) $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B \xrightarrow{\frac{d}{dx}} y' = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}} - \dfrac{A}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y' = 2\sin^{-1}x - A$

$\Rightarrow \sqrt{1-x^2}\, y'' - \dfrac{x}{\sqrt{1-x^2}}\, y' = \dfrac{2}{\sqrt{1-x^2}}$

(v) $l(a+bx) + \dfrac{y}{x} = l(x) \Rightarrow y = x\,l(x) - x\,l(bx+a)$

$\Rightarrow y' = l(x) + 1 - l(bx+a) - \dfrac{bx}{bx+a}$

$= l(x) - l(bx+a) + \dfrac{a}{bx+a}$

$\Rightarrow xy' = \overbrace{x\,l(x) - l(bx+a)}^{y} + \left(\dfrac{ax}{bx+a}\right)$

$(xy' - y) = \left(\dfrac{ax}{bx+a}\right)$

d/dx

$y'' = \dfrac{1}{x} - \dfrac{b}{bx+a} - \dfrac{ab}{(bx+a)^2}$

$x^2y'' = x - \dfrac{bx^2}{(bx+a)} - \dfrac{abx^2}{(bx+a)^2}$

$= \dfrac{\cancel{b^2x^3} + 2abx^2 + a^2x - \cancel{b^2x^3} - abx^2 - abx^2}{(bx+a)^2}$

$= \dfrac{a^2x}{(bx+a)^2}$

$\Rightarrow x^2y'' = (xy' - y)^2$

(vi) $y = kx^2 - 2k^2x + k^3 \xrightarrow{\frac{d}{dx}} y' = 2kx - 2k^2$

$\Rightarrow 2k^2 - 2xk + y' = 0$

$\Rightarrow k = \dfrac{x \pm \sqrt{x^2 - 2y'}}{2}$

$$y = \left(\frac{x+\sqrt{x^2-2y'}}{2}\right)\left(x^2 - x\left(x+\sqrt{x^2-2y'}\right) + \left(\frac{x+\sqrt{x^2-2y'}}{2}\right)^2\right)$$

$$= \left(\frac{x+\sqrt{x^2-2y'}}{2}\right)\left(\frac{\left(x+\sqrt{x^2-2y'}\right)^2}{4} - x\sqrt{x^2-2y'}\right)$$

$$= \frac{1}{8}\left(x+\sqrt{x^2-2y'}\right)\left(x+\sqrt{x^2-2y'}\right)^2$$

$$\Rightarrow y = \frac{y'}{4}\left(x - \sqrt{x^2-2y'}\right)$$

(vii) Let the parabola be $\overbrace{(y-c)^2 = a(x-b)}^{\text{Parameters}}$

fixed

$\Rightarrow y^2 - 2cy + c^2 = ax - ab$

$\frac{d}{dx} \Rightarrow 2yy' - 2cy' = a$

$\Rightarrow y - \dfrac{a}{2y'} = c$

$\frac{d}{dx} \Rightarrow y' + \dfrac{a}{2(y')^2}\, y'' = 0$

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→ Relⁿ b/w # Independent parameters & order of DE for a given family of curves

$$\boxed{\begin{pmatrix}\text{Order} \\ \text{of D.E}\end{pmatrix} = \begin{pmatrix}\text{\# Independent} \\ \text{parameters in eq}^n \\ \text{of family of curves}\end{pmatrix}}$$

eg –

		Order	
(i)	Family of Straight lines	2	(Pt. & Slope)
(ii)	Family of Circles	3	(2 coordinates of centre & Radius)
(iii)	General Parabola	4	(2 for Directrix, 2 for Focus)
(iv)	General Hyperbola or General Ellipse	5	(2 for Directrix, 2 for Focus, 1 for Eccentricity)

Q. Find order of D.E.

		Order
(i)	$y = c_1 s^2 x + c_2 c_{2x} + c_3$	2
(ii)	$y = (c_1+c_3)\, c_{(x+c_3)} - c_4 e^{(x+c_5)}$	3
(iii)	$y^2 = 2c(x+\sqrt{c})$	1
(iv)	Family of parabolas having fix directrix	2

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SOLUTION OF DE

(I) Variable - Separable Method

$$\frac{dy}{dx} = h(x,y)$$

$$\Rightarrow \quad f(x)\,dx = g(y)\,dy$$

$$\Rightarrow \quad \int f(x)\,dx = \int g(y)\,dy$$

Q. (i) $\sec^2(x)\tan y\,dx + \sec^2(y)\tan x\,dy = 0$

(ii) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

(iii) $\sqrt{1+x^2+y^2+x^2y^2} + (dy/dx)\,xy = 0$

(iv) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

A (i) $\int \frac{\sec^2(x)}{\tan x}\,dx = \int \frac{-\sec^2(y)}{\tan y}\,dy \Rightarrow \underline{l|\tan x| = -l|\tan y| + C}$

(ii) $\int e^y\,dy = \int (e^x + x^2)\,dx$

$$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$$

(iii) $\int \frac{y}{\sqrt{1+y^2}}\,dy = \int -\frac{dx}{x\sqrt{1+x^2}} \Rightarrow \sqrt{1+y^2} = l\left|\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right| + C$

(iv) $\int \frac{dy}{y-ay^2} = \int \frac{dx}{(x+a)} \Rightarrow \int \frac{ay+1-ay}{y(1-ay)}\,dy = l(x+a)$

$\Rightarrow l|y| - l|1-ay| = l|x+a| + C$

Q. (i) $dy/dx = s^2(x+3y) + 5$

(ii) $(x+y)^2(dy/dx) = a^2$

(iii) $(2x+3y-1)\,dx + (4x+6y-5)\,dy = 0$

(iv) $\frac{dy}{dx} = c(x+y) - s(x+y)$

A. (i) $u = x+3y$

$\Rightarrow \frac{du}{dx} = 1 + 3\frac{dy}{dx}$

$\Rightarrow \frac{1}{3}\frac{du}{dx} - \frac{1}{3} = s^2(u) + 5$

$\Rightarrow \frac{du}{dx} - 1 = 3s^2u + 15$

$\Rightarrow \int \frac{du}{3s^2u+16} = \int dx$

$\Rightarrow \int \frac{sec^2(u)\,du}{19t^2+16} = x$

$\Rightarrow \frac{1}{4\sqrt{19}}\, t^{-1}\left(\frac{\sqrt{19}\,t}{4}\right) = x + C$

(ii) $u = x+y$ $\Rightarrow (u^2)\left(\frac{du}{dx} - 1\right) = a^2$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = \frac{u^2+a^2}{u^2}$

$\Rightarrow \int\left(1 - \frac{a^2}{u^2+a^2}\right) du = \int dx$

$\Rightarrow \underline{(x+y) - a\, t^{-1}\left(\frac{x+y}{a}\right) = x + C}$

(iii) $u = 2x+3y$ $\Rightarrow (u-1) + \frac{(2u-5)}{-3}\left(\frac{du}{dx} - 2\right) = 0$

$\frac{du}{dx} = 2 + 3\frac{dy}{dx}$

$\Rightarrow (3u-3 \; -2u+10) + (2u-5)\frac{du}{dx} = 0$

$\Rightarrow \frac{2u-5}{u+7}\, du = -dx$

$\Rightarrow \int 2 - \left(\frac{19}{u+7}\right) du = \int -dx$

$\Rightarrow \underline{2 - 19\, l\,|2x+3y+7| = -x + C}$

(iv) $u = x+y$ $\Rightarrow \frac{du}{dx} - 1 = C_u - S_u$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \int \frac{du}{C_u - S_u + 1} = \int dx$

$\Rightarrow \int \frac{1+t^2}{1-\cancel{t^2}-2t+1+\cancel{t^2}}\, du = x$

$\Rightarrow x = \int \frac{d(t_{u/2})}{2 - 2t_{u/2}} = -l\left|\tan\left(\frac{x+y}{2}\right) - 1\right| + C$

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(II) Homogenous Eq^n

A D.E $P(x,y)\,dx + Q(x,y)\,dy = 0$

is called a homogenous eq^n if $P(x,y)$ & $Q(x,y)$ are homogenous fn^{n}s of same degree in x & y

To check if a D.E is homo., replace

$$x \rightarrow kx$$
$$y \rightarrow ky$$

if D.E remains same, then it is a homo D.E

To solve homo D.E, subs $\boxed{y = vx}$

& $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

This converts a homo D.E to variable separable D.E

Q. (i) $y\,dx + (2\sqrt{xy} - x)\,dy = 0$

(ii) $(x^2 + y^2)\,dx = 2xy\,dy$

(iii) $2\,dy/dx = y/x + y^2/x^2$

(iv) $(1 + 2e^{x/y})\,dx + 2e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$

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A. (i) $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow vx + (2\sqrt{v}x - x)\left(v + x\frac{dv}{dx}\right) = 0$

$\Rightarrow \cancel{v} + (2v^{3/2} - \cancel{v}) + x(2v^{1/2} - 1)\frac{dv}{dx} = 0$

$\Rightarrow \int -\frac{1}{x}\,dx = \int \frac{1}{v} - \frac{1}{2v^{3/2}}\,dv$

$\Rightarrow -l|x| + C = l|v| + \frac{1}{\sqrt{v}}$

$\Rightarrow l\left|\frac{y}{x}\right| + \sqrt{\frac{x}{y}} + l|x| = C$

(ii) $y = vx$

$\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$

$\Rightarrow (v^2+1)x^2 - 2vx^2\left(v + x\frac{dv}{dx}\right) = 0$

$\Rightarrow (1 - v^2) = 2vx\frac{dv}{dx}$

$\Rightarrow \int -\frac{dx}{x} = \int \frac{2v\,dv}{(v^2-1)}$

$\Rightarrow -l|x| = l|v^2-1| + C$

$\Rightarrow l\left|\frac{y^2}{x^2} - 1\right| + l|x| + C = 0$

(iii) $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow 2\left(v + x\frac{dv}{dx}\right) = v + v^2$

$\Rightarrow \int \frac{dv}{v^2 - v} = \frac{1}{2}\int \frac{dx}{x}$

$\Rightarrow l|v-1| - l|v| = \frac{l|x|}{2} + C$

$\Rightarrow l\left|\frac{y}{x} - 1\right| - l\left|\frac{y}{x}\right| = \frac{1}{2}l|x| + C$

(iv) $x = vy$

$\frac{dx}{dy} = v + y\frac{dv}{dy}$

$\Rightarrow (1+2e^{v})\left(v + y\frac{dv}{dy}\right) + 2e^{v}(1-v) = 0$

$\Rightarrow v + y\frac{dv}{dy} + \cancel{2e^{v}\cdot v} + 2e^{v}y\frac{dv}{dy} + 2e^{v} - \cancel{2e^{v}v} = 0$

$\Rightarrow y(2e^{v}+1)\frac{dv}{dy} = -(2e^{v}+v)$

$\Rightarrow \int\left(\frac{2e^{v}+1}{2e^{v}+v}\right)dv = -\int\frac{dy}{y}$

$\Rightarrow l|2e^{v}+v| = -l|y| + C$

$\Rightarrow \underline{l|2e^{x/y} + x/y| = -l|y| + C}$

→ Reducible to Homogenous D.E

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

(i) $a_1b_2 \neq a_2b_1$

Method : $x = X + h$, $y = Y + k$ $\Rightarrow \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \Rightarrow \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$

Choose h, k s.t $a_1h + b_1k + c_1 = 0$
& $a_2h + b_2k + c_2 = 0$

$\left\{\text{Homogenity of } \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \text{ is being lost due to the const. term. This method has been adopted to remove the consts \& make the eq}^n \text{ homo.}\right\}$

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Q. (i) $(2x-y+4)\,dy + (x-2y+5)\,dx = 0$

(ii) $\frac{dy}{dx} = \frac{1-3y-3x}{1+x+y}$

A. (i) $X = x+h$ $\Rightarrow$ $2h-k+4=0$; $h=-1$
$Y = y+k$ & $h-2k+5=0$; $k=2$

$\Rightarrow (2X-Y)\,dY + (X-2Y)\,dX = 0$

$Y = vX$ $\Rightarrow (2-v)\left(v + X\frac{dv}{dX}\right) + (1-2v) = 0$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$ $\Rightarrow \cancel{2v} - v^2 + (2-v)X\frac{dv}{dX} + 1 - \cancel{2v} = 0$

$\Rightarrow \int\left(\frac{v-2}{v^2-1}\right)dv = \int -\frac{dX}{X}$

$\Rightarrow l|v+1| - \frac{1}{2}\,l\left|\frac{v-1}{v+1}\right| = -l|X|$

$\Rightarrow l\left|\frac{y+2}{x-1}+1\right| - \frac{1}{2}\,l\left|\frac{(y+2)-(x-1)}{(y+2)+(x-1)}\right| = -l|x-1|$

(ii) Since $a_1b_2 = a_2b_1$, we cannot use the method.
We will instead use sub^n

$u = x+y$ $\Rightarrow \left(\frac{du}{dx} - 1\right) = \frac{1-3u}{1+u}$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$ $\Rightarrow du/dx = 2\left(\frac{1-u}{1+u}\right)$

$\Rightarrow \int \frac{u+1}{u-1}\,du = \int -2\,dx$ $\Rightarrow (x+y) + 2l|x+y-1| = -2x$

(II) First Order Linear D.E

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

or $$\frac{dx}{dy} + q(y)\,x = H(y)$$

For solving these eq^n s, we use integrating factor

$$\boxed{F_i = e^{\int P(x)\,dx}} \quad \text{or} \quad e^{\int q(y)\,dy}$$

Sol^n of the eq^n is

$$\boxed{y\left(e^{\int P(x)\,dx}\right) = \int Q(x)\cdot e^{\int P(x)\,dx}\,dx + C}$$

Q. (i) $dy/dx + 2y = cx$ (ii) $dy/dx + y/x = \ell|x|$

(iii) $\dfrac{dy}{dx} = \dfrac{y}{2y\ell|y| + y - x}$ (iv) $\dfrac{dy}{dx} + y\varphi'(x) = \varphi(x)\varphi'(x)$

A. (i) $F_i = e^{\int 2\,dx} = e^{2x} \Rightarrow ye^{2x} = \int \underbrace{e^{2x}\,cx}_{I}\,dx$

$\Rightarrow y = \dfrac{s+2c}{5} + \dfrac{C}{e^{2x}}$

$I = e^{2x}sx + 2e^{2x}cx - 4I$

$\Rightarrow I = \dfrac{e^{2x}(s+2c)}{5}$

D	I
e^{2x}	cx
$2e^{2x}$	sx
$4e^{2x}$	$-cx$

(ii) $F_i = e^{\int \frac{dx}{x}} = e^{l(x)} = x \Rightarrow yx = \int x\,l(x)\,dx$

$\Rightarrow yx = \frac{x^2 l(x)}{2} - \int \frac{x}{2}\,dx$

D	I
$l(x)$	x
$\frac{1}{x}$	$\frac{x^2}{2}$

$\Rightarrow y = \frac{x\,l(x)}{2} - \frac{x}{4} + \frac{C}{x}$

(iii) $\frac{dx}{dy} = 2l(y) + 1 - \frac{x}{y} \Rightarrow \frac{dx}{dy} + \frac{x}{y} = 2l(y)+1$

$F_i = e^{\int \frac{1}{y} dy} = e^{l(y)} = y \Rightarrow xy = \int (2l(y)+1)\,y\,dy$

$\Rightarrow xy = \frac{y^2}{2}(2l(y)+1) - \int y\,dy$

$\Rightarrow x = \frac{y}{2}(2l(y)+1) - \frac{y}{2} + \frac{C}{y}$

D	I
$2l(y)+1$	y
$\frac{2}{y}$	$\frac{y^2}{2}$

(iv) $F_i = e^{\int \varphi'(x)\,dx} = e^{\varphi(x)}$

$\Rightarrow y e^{\varphi(x)} = \int e^{\varphi(x)}\,\underline{\varphi(x)\,\varphi'(x)}\,dx$

D	I
$\varphi(x)$	$e^{\varphi(x)}\varphi'(x)$
$\varphi'(x)$	$e^{\varphi(x)}$

$I = e^{\varphi(x)}\varphi(x) - I$

$\Rightarrow I = \frac{1}{2} e^{\varphi(x)}\varphi(x)$

$\Rightarrow y e^{\varphi(x)} = \frac{1}{2} e^{\varphi(x)}\varphi(x) + C$

→ Extended linear D.E (Bernoulli's Eq^n)

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n}$$

$$\Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} + P(x)\left(\frac{1}{y^{n-1}}\right) = Q(x)$$

Sub. $u = y^{-(n-1)}$ $\Rightarrow$ D.E would be converted to standard L.D.E

Q (i) $(y\ell(x) - 1)\, y\, dx = x\, dy$ (ii) $dy/dx + xy = xy^2$

(iii) $\dfrac{dy}{dx} = \dfrac{y\varphi'(x) - y^2}{\varphi(x)}$

A. (i) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{\ell(x)}{x}\, y^2 \Rightarrow \dfrac{1}{y^2}\dfrac{dy}{dx} + \left(\dfrac{1}{y}\right)\left(\dfrac{1}{x}\right) = \dfrac{\ell(x)}{x}$

$u = y^{-1}$

$\Rightarrow \dfrac{du}{dx} = -\left(\dfrac{dy}{dx}\right)\left(\dfrac{1}{y^2}\right) \Rightarrow \dfrac{du}{dx} - \dfrac{u}{x} = \dfrac{-\ell(x)}{x}$

Fi $= e^{\int \frac{-dx}{x}} = e^{-\ln|x|} = \left(\dfrac{1}{x}\right) \Rightarrow \dfrac{u}{x} = \displaystyle\int \dfrac{-\ell(x)}{x^2}\, dx$

$\Rightarrow \dfrac{u}{x} = \dfrac{\ell(x)}{x} - \displaystyle\int x^{-2}\, dx$

D	I
$\ell(x)$	$-\frac{1}{x^2}$
$\frac{1}{x}$	$\frac{1}{x}$

$\Rightarrow \dfrac{1}{xy} = \dfrac{\ell(x)}{x} + \dfrac{1}{x} + C$

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(ii) $\frac{1}{y^2}\left(\frac{dy}{dx}\right) + \left(\frac{x}{y}\right) = x$

$u = \frac{1}{y}$ $\Rightarrow \frac{du}{dx} - xu = -x$

$\frac{du}{dx} = -\frac{1}{y^2}\left(\frac{dy}{dx}\right)$ $F_i = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}$

$\Rightarrow u\left(e^{-x^2/2}\right) = \int e^{-x^2/2}(-x)\,dx$

$\Rightarrow \frac{1}{ye^{x^2/2}} = e^{-x^2/2} + C$

(iii) $\frac{dy}{dx} - \frac{\varphi'(x)}{\varphi(x)} y = -\frac{1}{\varphi(x)} y^2$

$\Rightarrow -\frac{1}{y^2}\left(\frac{dy}{dx}\right) + \frac{\varphi'(x)}{\varphi(x)}\left(\frac{1}{y}\right) = -\frac{1}{\varphi(x)}$

$u = \frac{1}{y}$ $\Rightarrow \frac{du}{dx} + \frac{\varphi'(x)}{\varphi(x)} u = -\frac{1}{\varphi(x)}$

$\frac{du}{dx} = -\frac{1}{y^2}\left(\frac{dy}{dx}\right)$

$F_i = e^{\int \frac{\varphi'(x)}{\varphi(x)}dx} = e^{\ln(\varphi(x))} = \varphi(x)$

$\Rightarrow y\varphi(x) = \int -\frac{\varphi(x)}{\varphi(x)}\,dx$

$\Rightarrow y\varphi(x) = -x + C$

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→ Special form (Reducible to L.D.E)

$$f'(y)\frac{dy}{dx} + P(x)\, f(y) = Q(x)$$

sub $u = f(y)$ to get L.D.E

Q. (i) $\sec^2(y)\, dy/dx + 2x \tan y = x^3$

(ii) $\sin(y)\, dy/dx = \cos y(1 - x\cos y)$

A. (i) $u = \tan y$ $\Rightarrow$ $\frac{du}{dx} + 2xu = x^3$

$\frac{du}{dx} = \sec^2(y)\left(\frac{dy}{dx}\right)$

$F_i = e^{\int 2x\,dx} = e^{x^2}$

$\Rightarrow u e^{x^2} = \int \underbrace{e^{x^2} x^3}\, dx$

D	I
$\frac{x^2}{2}$	$2x e^{x^2}$
$\frac{x^3}{6}$	e^{x^2}

$\Rightarrow \tan y\, e^{x^2} = \frac{3}{7} x^2 e^{x^2} + C$

$I = \frac{x^2 e^{x^2}}{2} - \frac{I}{6}$

$\Rightarrow I = \frac{3}{7} x^2 e^{x^2}$

(ii) $\tan y \left(\frac{dy}{dx}\right) + \cos(y)\, x = 1$

$\Rightarrow \tan y \sec(y)\left(\frac{dy}{dx}\right) - \sec(y) = -x$

$u = \sec(y)$ $\Rightarrow$ $\frac{du}{dx} - u = -x$ $F_i = e^{\int -1\,dx} = e^{-x}$

$\frac{du}{dx} = \sec(y) \tan y \frac{dy}{dx}$

$\Rightarrow \frac{u}{e^x} = \int -e^{-x} x\, dx$

D	I
x	$-e^{-x}$
1	e^{-x}

$\Rightarrow \frac{\sec(y)}{e^x} = xe^{-x} - \int e^{-x} dx = (x+1)e^{-x}$

$\Rightarrow \sec(y) = (x+1) + Ce^x$

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(IV) Exact D.E

An eqn of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Formulae :-

1. $x\,dy + y\,dx = d(xy)$

2. $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$

3. $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

4. $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

5. $\frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left(l\left(\frac{y}{x}\right)\right)$

6. $\frac{y\,dx - x\,dy}{xy} = d\left(l\left(\frac{x}{y}\right)\right)$

7. $\frac{x\,dy - y\,dx}{x^2+y^2} = \frac{\left(\frac{x\,dy - y\,dx}{x^2}\right)}{1+\left(\frac{y^2}{x^2}\right)} = \frac{d(y/x)}{1+\left(\frac{y}{x}\right)^2} = d\left(t^{-1}\left(\frac{y}{x}\right)\right)$

8. $\frac{dx + dy}{x+y} = d(l(x+y))$

9. $\frac{x\,dy + y\,dx}{xy} = d(l(xy))$

10. $\frac{x\,dx + y\,dy}{x^2+y^2} = d\left(\frac{1}{2}l(x^2+y^2)\right)$

11. $\frac{x\,dy + y\,dx}{x^2y^2} = d\left(\frac{-1}{xy}\right)$

12. $d(e^y/x) = \frac{xe^y\,dy - e^y\,dx}{x^2}$

13. $d(e^x/y) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

14. $d(x^m y^n) = x^{(m-1)}\,y^{(n-1)}\,(my\,dx + nx\,dy)$

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Q (i) $(x^2 - ay)dx + (y^2 - ax)dy = 0$

(ii) $x dx + y dy = x dy - y dx$

(iii) $2x\, l(y)\, dx + \left(\frac{x^2}{y} + 3y^2\right) dy = 0$

(iv) $\dfrac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

(v) $\dfrac{y + sx\,c^2(xy)}{c^2(xy)}\, dx + \left(\dfrac{x}{c^2_{xy}} + sy\right) dy = 0$

(vi) $x^2 dy - y^2 dx + xy^2(x-y)\, dy = 0$

☆ (vii) $y\, dx - x\, dy + xy^2 dx = 0$

(viii) $x\, dy\,(y^2 e^{xy} + e^{x/y}) = y\, dx\,(e^{x/y} - y^2 e^{xy})$

A. (i) $\int x^2 dx + \int y^2 dy = \int a(y dx + x dy) = \int a\, d(xy)$

$\Rightarrow x^3 + y^3 = 3axy + C$

(ii) $d(xy) = xy\, d\left(l\left(\frac{y}{x}\right)\right) \Rightarrow \int \frac{d(xy)}{xy} = \int d\left(l\left(\frac{y}{x}\right)\right)$

$\Rightarrow l(xy) = l\left(\frac{y}{x}\right) + C$

(iii) $2x\, l(y)\, dx + \frac{x^2}{y} dy = -3y^2 dy \Rightarrow \int d(x^2 l(y)) = \int -d(y^3)$

$\Rightarrow x^2 l(y) = -y^3 + C$

(iv) $\dfrac{x\,dx + y\,dy}{(x^2+y^2)^2} = \dfrac{y\,dx - x\,dy}{x^2} \Rightarrow \dfrac{-1}{2}\, d\left(\dfrac{1}{x^2+y^2}\right) = +d\left(\dfrac{y}{x}\right)$

$\Rightarrow \dfrac{1}{2(x^2+y^2)} = \dfrac{y}{x} + C$

(v) $\underbrace{(x\,dy + y\,dx)}_{c^2(xy)} + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow \int \sec^2(xy)\, d(xy) + \int \sin x\, dx + \int \sin y\, dy = 0$

$\Rightarrow \tan(xy) = \cos x + \cos y + C$

* (vi) $x^2 + xy^2(x-y) = y^2\left(\dfrac{dx}{dy}\right) \Rightarrow -x^2(1+y^2) - xy^3 = y^2\left(\dfrac{dx}{dy}\right)$

$u = \dfrac{1}{x}$

$\dfrac{du}{dy} = -\dfrac{1}{x^2}\left(\dfrac{dx}{dy}\right)$

$\Rightarrow \left(\dfrac{1}{x^2}\right)\left(\dfrac{dx}{dy}\right) + (y)\left(\dfrac{1}{x}\right) = 1 + \dfrac{1}{y^2} \Rightarrow \dfrac{du}{dy} - yu = 1 + \dfrac{1}{y^2}$

$\Rightarrow -\dfrac{e^{-y^2/2}}{x} = \int e^{-y^2/2}\left(1 + \dfrac{1}{y^2}\right) dy = \int e^{-y^2/2}\left(\left(\dfrac{1}{y}\right)(-y) + \dfrac{1}{y^2}\right) = C + \dfrac{e^{-y^2/2}}{y}$

(vii) $\int \underbrace{\dfrac{x\,dy - y\,dx}{y^2}}_{-d\left(\frac{x}{y}\right)} = \int x\,dx \Rightarrow \dfrac{x^2}{2} + \dfrac{x}{y} + C = 0$

$\Rightarrow \dfrac{1}{x} - \dfrac{1}{y} = Ce^{y^2/2}$

(viii) $(xy^2\,dy + y^2\,dx)\, e^{xy} + (x\,dy - y\,dx)\, e^{x/y} = 0$

$\Rightarrow \int e^{xy}\, d(xy) = \int e^{x/y}\, d\left(\dfrac{x}{y}\right)$

$\Rightarrow e^{xy} = e^{x/y} + C$

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family.

eg – Each line passing through origin i.e $y = mx$ is orthogonal trajectory of FoC $x^2+y^2=k^2$

- Procedure to find OT :-

(i) Let $f(x,y,c)=0$

(ii) Find D.E for $f(x,y,c)$ by diff. wrt. x

(iii) Subs $\frac{dy}{dx} \longrightarrow \left(\frac{-dx}{dy}\right)$ in the D.E of FoC

The D.E obtained represents the D.E of OT

(iv) Solve the obtained D.E to find OT.

Q Find OT.

(i) $xy=c$ (ii) $y=cx^2$ (iii) $x^2+y^2-2cx=0$

A. (i) $x\frac{dy}{dx}+y=0 \longrightarrow x\left(\frac{-dx}{dy}\right)+y=0 \Rightarrow \int y\,dy=\int x\,dx$

$\Rightarrow \underline{x^2-y^2=C}$

(ii) $\frac{y}{x^2}=c \Rightarrow \frac{x^2dy - 2xy\,dx}{x^2}=0$

$\Rightarrow x\left(\frac{dy}{dx}\right)-2y=0 \longrightarrow x\left(\frac{-dx}{dy}\right)-2y=0 \Rightarrow \int x\,dx+\int 2y\,dy=0$

$\Rightarrow \underline{x^2+2y^2=C}$

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(iii) $x + \frac{y^2}{x} = 2c \Rightarrow 1 + \frac{2xy\frac{dy}{dx} - y^2}{x^2} = 0$

$$\Rightarrow (x^2-y^2) + 2xy\frac{dy}{dx} = 0$$

$$\downarrow$$

$$(x^2-y^2) + 2xy\left(\frac{-dx}{dy}\right) = 0$$

$$\Rightarrow (x^2-y^2)\,dy = 2xy\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \Rightarrow v + x\frac{dv}{dx} = \frac{2v}{(1-v^2)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v + v^3}{(1-v^2)}$$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \frac{1-v^2}{v^3+v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \frac{2((1+v^2)-v^2)}{v(1+v^2)} - \frac{1}{v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{v} - \frac{2v}{(1+v^2)}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow l|v| - l|1+v^2| = l|x| + C$$

$$\Rightarrow l\left|\frac{y}{x}\right| - l\left|1+\frac{y^2}{x^2}\right| = l(x) + C$$